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The stationary product measure of multi-species ASEP with attachment and detachment

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Abstract
We consider the multi-species asymmetric simple exclusion process (ASEP) with attachment and detachment on an open chain. We find a necessary and sufficient condition for the model to have the stationary state as a product of scalars. First, we obtain a necessary condition on parameters. Next, we show that it is also a sufficient condition. We give the condition in some restricted cases. The single-site weight can be written in a determinant form in the case where the single-site weights are homogeneous.

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1. Introduction

Although non-equilibrium is one of the most attractive themes of physics and has been studied intensively, it is fair to say that a complete coherence has not been obtained. One way to study the non-equilibrium is to treat specific models. In recent years, models formulated as stochastic processes of many-particle systems including driven lattice gas systems have been studied intensively [1]. In the driven lattice gas systems, a large number of degrees of freedom are collected into a few effective variables that follow some probabilistic rules.

The asymmetric simple exclusion process (ASEP) in one dimension is such a model. They are used in modeling the real phenomena, such as traffic flow, molecular biology and so on [2]. The ASEP was proposed by C T MacDonald *et al* in [3] to formulate the dynamics of ribosomes along the messenger RNA. The discovery of the matrix product stationary state of ASEP with injection and extraction at the left and right ends was an important step which involves an exact derivation of boundary-induced phase transition [4]. The ASEP is also exactly solvable by means of the Bethe ansatz [5–8]. The two-species ASEP, in which the numbers of first and second kinds of particles are conserved in the bulk, also exhibit interesting phenomena, such as spontaneous symmetry breaking, phase separations and condensation [9]. The stationary-state solutions of some multi-species ASEPs (including the two-species ASEP) can be obtained as the matrix product form [10]. The ASEP and generalized ASEP models attracted much attention for their solvability, various interesting phenomena, valuable insight into non-equilibrium systems and applicability in spite of these simple settings.

In the living cells, many kinds of motor proteins move along the filament [11]. The feature of the dynamics of the motor proteins which the standard ASEP does not have is attachment and detachment in the bulk. The ASEP on an open chain with particle attachment and detachment allowed in the bulk was studied in [12, 13]. Its two-species generalizations are also studied in [14]. In this paper, we introduce the multi-species ASEP on an open chain with attachment and detachment of particles. Namely, we treat the multi-species particle system in which the following two types of events occur: (i) particles in nearest-neighbor sites exchange their positions with a specific rate. (ii) A particle at each site is detached and another kind of particle is attached simultaneously with a specific rate regardless of their neighbors. Each site is occupied by at most one particle. We define L and N as the number of site and the number of state at each site, respectively.

The main result of this paper is the explicit expression of the necessary and sufficient condition that the probability of finding a configuration $(\tau_1, \ldots, \tau_L)$ in the stationary state can be written in the following product form:

$$P(\tau_1, \ldots, \tau_L) = \frac{1}{Z} \prod_{1 \leqslant j \leqslant L} D_j(\tau_j), \tag{1}$$

where the single-site weight $D_j(\tau)$ is a scalar and Z is the normalizing factor defined by

$$Z = \prod_{1 \leqslant j \leqslant L} \sum_{1 \leqslant \tau \leqslant N} D_j(\tau). \tag{2}$$

If a model has the stationary product measure, we can calculate the densities, the currents and arbitrary equal-time correlation functions in the stationary state. The density of the nth kind of particles, that is the one-point function, is calculated as

$$\begin{aligned} \langle \chi_{\tau_j = n} \rangle &= \frac{1}{Z} \sum_{1 \leqslant \tau_i \leqslant N} P(\tau_1, \ldots, \tau_{j-1}, n, \tau_{j+1}, \ldots \tau_L) \\ &= \frac{D_j(n)}{D_j(1) + \cdots + D_j(N)}, \end{aligned} \tag{3}$$

where $\chi_{\mathrm{eq}} = 1$ if eq is true and $\chi_{\mathrm{eq}} = 0$ if eq is false. An arbitrary equal-time m-point correlation function can be obtained as a product of m one-point functions:

$$\langle \chi_{\tau_{k_1} = n_1} \cdots \chi_{\tau_{k_m} = n_m} \rangle = \langle \chi_{\tau_{k_1} = n_1} \rangle \cdots \langle \chi_{\tau_{k_m} = n_m} \rangle. \tag{4}$$

It is well known that the zero-range process (ZRP) has the stationary product measure [15]. The ZRP on a ring is mapped to the ASEP on a ring by considering particles in the ZRP as vacancies in the ASEP and sites in the ZRP as occupied sites in the ASEP. The ZRP with open boundaries is studied in [16]. However it cannot be mapped to the open ASEP because non-conservation of particles at both the ends of the ZRP makes non-conservation of the length of the chain of the ASEP.

The content of this paper is arranged as follows. In section 2, we formulate the model and impose a necessary condition for the stationary product measure. In section 3, we prove that the condition obtained in section 2 is also a sufficient condition. In section 4, we show examples with parameters restricted. The conclusion and discussion are given in section 5.

2. Model and necessary condition

Let us consider a system on an L-site open chain with the following rules (see figure 1). The number of states per site is N and the state $\tau = 1$ is regarded as vacancy.

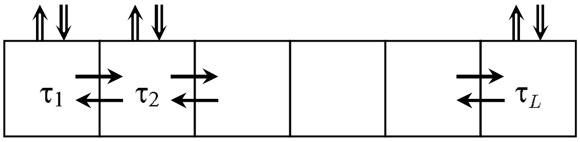

Figure 1. Multi-species ASEP with attachment and detachment on an open chain.

The particles exchange their positions at each bond between the jth and the $(j+1)$th sites as

$$xy \quad \Longrightarrow \quad yx \qquad \text{with a rate} \quad p_j(xy \to yx) \tag{5}$$

for $1 \leqslant x, y \leqslant N, x \neq y$. Detachment of x and attachment of y take place simultaneously at each jth site as

$$x \quad \Longrightarrow \quad y \qquad \text{with a rate} \quad \omega_j(x \to y) \tag{6}$$

for $1 \leqslant x, y \leqslant N, x \neq y$. This can be regarded as transformation from x to y, annihilation of x if $y = 1$ or creation of y if $x = 1$. Let us set $p_j(xx \to xx) = \omega_j(x \to x) = p_0(xy \to yx) = p_L(xy \to yx) = 0$ to prevent the equations in what follows from being complicated. Note that the rates $p_j(xy \to yx)$ and $\omega_j(x \to y)$ depend on the site number j.

Let $P(\tau_1, \ldots, \tau_L)$ be the probability that a configuration $(\tau_1, \ldots, \tau_L)$ is found. It is characterized by the following master equation:

$$-\frac{\mathrm{d}}{\mathrm{d}t} P(\tau_1, \ldots, \tau_L) = Q[P](\tau_1, \ldots, \tau_L), \tag{7}$$

where Q is defined by

$$\begin{aligned} Q[P](\tau_1, \ldots, \tau_L) := &\sum_{1 \leqslant j \leqslant L-1} p_j(\tau_j \tau_{j+1} \to \tau_{j+1}\tau_j) P(\tau_1, \ldots, \tau_L) \\ &- \sum_{1 \leqslant j \leqslant L-1} p_j(\tau_{j+1}\tau_j \to \tau_j \tau_{j+1}) P(\tau_1, \ldots, \tau_{j-1}, \tau_{j+1}, \tau_j, \tau_{j+2}, \ldots, \tau_L) \\ &+ \sum_{1 \leqslant j \leqslant L} \sum_{1 \leqslant z \leqslant N} \omega_j(\tau_j \to z) P(\tau_1, \ldots, \tau_L) \\ &- \sum_{1 \leqslant j \leqslant L} \sum_{1 \leqslant z \leqslant N} \omega_j(z \to \tau_j) P(\tau_1, \ldots, \tau_{j-1}, z, \tau_{j+1}, \ldots, \tau_L). \end{aligned} \tag{8}$$

The stationary-state solution is defined by

$$Q[P](\tau_1, \ldots, \tau_L) = 0 \tag{9}$$

for every configuration $(\tau_1, \ldots, \tau_L)$.

The form of the stationary-state solution is nontrivial if the parameters are chosen arbitrarily. The question is whether there is a parameter region where the model has the stationary product measure. A systematic way to obtain a necessary condition for the existence of finite-dimensional matrix product stationary states for multi-state reaction–diffusion processes with open boundaries was introduced in [17]. However, it is not applicable to the model in this paper because of its inhomogeneity. Let us impose a necessary condition on the parameters in another way, assuming that the model has the stationary product measure (1).

First, we calculate the element of $Q[P]$ corresponding to the configuration $(x,\ldots,x)$ for $1 \leqslant x \leqslant N$. This element must be 0 if P is a stationary-state solution. The calculation is very simple:

$$Q[P](x,\ldots,x)/D_1(x)\cdots D_L(x) = \sum_{1\leqslant k\leqslant L}\sum_{1\leqslant z\leqslant N}\left(\omega_k(x\to z) - \omega_k(z\to x)\frac{D_k(z)}{D_k(x)}\right) = 0. \tag{10}$$

Next, we calculate the element of $Q[P]$ corresponding to the configuration $(x,\ldots,x,\overset{j\text{th}}{y},x,\ldots,x)$ for $1\leqslant x,y\leqslant N$ $(x\neq y)$. This element also must be 0:

$$\begin{aligned}
&Q[P](x,\ldots,x,y,x,\ldots,x)/D_1(x)\cdots D_{j-1}(x)D_j(y)D_{j+1}(x)\cdots D_L(x)\\
&\quad= p_{j-1}(xy\to yx) + p_j(yx\to xy) - p_{j-1}(yx\to xy)\frac{D_{j-1}(y)D_j(x)}{D_{j-1}(x)D_j(y)}\\
&\qquad - p_j(xy\to yx)\frac{D_j(x)D_{j+1}(y)}{D_j(y)D_{j+1}(x)}\\
&\qquad + \sum_{\substack{1\leqslant k\leqslant L\\ k\neq j}}\sum_{1\leqslant z\leqslant N}\omega_k(x\to z) + \sum_{1\leqslant z\leqslant N}\omega_j(y\to z)\\
&\qquad - \sum_{\substack{1\leqslant k\leqslant L\\ k\neq j}}\sum_{1\leqslant z\leqslant N}\omega_k(z\to x)\frac{D_k(z)}{D_k(x)} - \sum_{1\leqslant z\leqslant N}\omega_j(z\to y)\frac{D_j(z)}{D_j(y)}\\
&\quad= 0.
\end{aligned} \tag{11}$$

Using (10), we can simplify this equation as

$$u_{j-1}(xy) + u_j(yx) = v_j(x) - v_j(y), \tag{12}$$

where

$$u_j(xy) = p_j(xy\to yx) - p_j(yx\to xy)\frac{D_j(y)D_{j+1}(x)}{D_j(x)D_{j+1}(y)}, \tag{13}$$

$$v_j(x) = \sum_{1\leqslant z\leqslant N}\omega_j(x\to z) - \sum_{1\leqslant z\leqslant N}\omega_j(z\to x)\frac{D_j(z)}{D_j(x)}. \tag{14}$$

Using the relation (12) and one with x and y replaced with each other, we find

$$u_j(xy) = -\sum_{1\leqslant k\leqslant j}(v_k(x) - v_k(y)), \tag{15}$$

$$u_j(yx) = -\sum_{1\leqslant k\leqslant j}(v_k(y) - v_k(x)). \tag{16}$$

The calculations above give a parameter space

$$\begin{aligned}
&\sum_{1\leqslant k\leqslant L} v_k(x) = 0,\ u_j(xy) = -\sum_{1\leqslant k\leqslant j}(v_k(x) - v_k(y))\\
&\qquad\text{for}\quad 1\leqslant x,y\leqslant N,\ x\neq y,\ 1\leqslant j\leqslant L-1,
\end{aligned} \tag{17}$$

which is the necessary condition for the model to have the stationary product measure. Although we show some examples in section 4, let us discuss here the second equation of (17). In oder for parameters satisfying both (15) and (16) to exist, at least one of the following two conditions must be satisfied:

$$\frac{D_j(y)D_{j+1}(x)}{D_j(x)D_{j+1}(y)} = 1, \tag{18}$$

$$u_j(xy) = -\sum_{1\leqslant k\leqslant j} (v_k(x) - v_k(y)) = 0. \tag{19}$$

The first choice (18) means that the single-site weights are locally homogeneous. In section 4, we study the condition in the case where the single-site weights are globally homogeneous. The second choice (19) provides

$$p_j(xy \to yx)D_j(x)D_{j+1}(y) = p_j(yx \to xy)D_j(y)D_{j+1}(x) \tag{20}$$

and means that the dynamics of the exchange of particles locally satisfies the detailed balance condition.

3. Sufficient condition

It is natural for us to have a question whether the model under the constraint (17) has the stationary product measure (1), that is, whether the condition (17) is also a sufficient condition for the stationary product measure. The answer is true. Defining $\hat{P}$ as the product form (1), we can check that

$$Q[\hat{P}] = 0 \tag{21}$$

under the constraint (17).

To prove this we prepare the following formula:

$$Q_i[\hat{P}](\tau_1,\ldots,\tau_L) = D_1(\tau_1)\cdots D_L(\tau_L) \sum_{1\leqslant\ell\leqslant i} v_\ell(\tau_{i+1}) \tag{22}$$

for $1 \leqslant i \leqslant L-1$, where Q_i is defined by

$$\begin{aligned} Q_i[P](\tau_1,\ldots,\tau_L) :=& \sum_{1\leqslant j\leqslant i} p_j(\tau_j\tau_{j+1} \to \tau_{j+1}\tau_j)P(\tau_1,\ldots,\tau_L) \\ &- \sum_{1\leqslant j\leqslant i} p_j(\tau_{j+1}\tau_j \to \tau_j\tau_{j+1})P(\tau_1,\ldots,\tau_{j-1},\tau_{j+1},\tau_j,\tau_{j+2},\ldots,\tau_L) \\ &+ \sum_{1\leqslant j\leqslant i}\sum_{1\leqslant z\leqslant N} \omega_j(\tau_j \to z)P(\tau_1,\ldots,\tau_L) \\ &- \sum_{1\leqslant j\leqslant i}\sum_{1\leqslant z\leqslant N} \omega_j(z \to \tau_j)P(\tau_1,\ldots,\tau_{j-1},z,\tau_{j+1},\ldots,\tau_L). \end{aligned} \tag{23}$$

We can prove this formula recursively under the constraint (17) (see the appendix).

Using this formula for $i = L-1$, we find

$$\begin{aligned} Q[\hat{P}](\tau_1,\ldots,\tau_L) =& Q_{L-1}[\hat{P}](\tau_1,\ldots,\tau_L) \\ &+ \sum_{1\leqslant z\leqslant N} \omega_L(\tau_L \to z)\hat{P}(\tau_1,\ldots,\tau_L) - \sum_{1\leqslant z\leqslant N} \omega_L(z \to \tau_L)\hat{P}(\tau_1,\ldots,\tau_{L-1},z) \\ &= D_1(\tau_1)\cdots D_L(\tau_L) \sum_{1\leqslant\ell\leqslant L-1} v_\ell(\tau_L) + D_1(\tau_1)\cdots D_L(\tau_L)v_L(\tau_L) \\ &= D_1(\tau_1)\cdots D_L(\tau_L) \sum_{1\leqslant\ell\leqslant L} v_\ell(\tau_L) \\ &= 0. \end{aligned} \tag{24}$$

Thus, we have proved that the necessary condition (17) is also a sufficient condition that the stationary-state solution can be written as the product of scalars (1).

4. Some restricted cases

4.1. Case of homogeneous D

In this subsection, we investigate the necessary and sufficient condition (17) in the case where the single-site weights are homogeneous: $D_j(x) = D(x)$. The first condition (10) is a little simplified as

$$D(x) \sum_{1\leqslant k\leqslant L} \sum_{1\leqslant z\leqslant N} \omega_k(x \to z) - \sum_{1\leqslant k\leqslant L} \sum_{1\leqslant z\leqslant N} D(z)\omega_k(z \to x) = 0. \tag{25}$$

This can be rewritten as

$$M \begin{pmatrix} D(1) \\ \vdots \\ D(N) \end{pmatrix} = 0 \tag{26}$$

with an $N \times N$ matrix M whose diagonal elements are

$$(M)_{aa} = \sum_{1\leqslant z\leqslant N} \sum_{1\leqslant k\leqslant L} \omega_k(a \to z), \tag{27}$$

and off-diagonal elements are

$$(M)_{ab} = -\sum_{1\leqslant k\leqslant L} \omega_k(b \to a) \quad (a \neq b). \tag{28}$$

Because the sum over columns of M vanishes, $D(x)$ has the following determinant form up to constant factors:

$$D(x) = \det M^{(x)}, \tag{29}$$

where $M^{(x)}$ is an $(N-1)\times(N-1)$ matrix defined by deleting the xth row and the xth column from the matrix M. Note that (29) is valid whenever the model has the stationary product measure with homogeneous single-site weights. It is interesting that the matrix M generates a process of N-state system on one site, where the transformation from b to a occurs with a rate $(M)_{ab}$ and the (relative) probability in the stationary state is given by (29). The second condition (15) becomes

$$\begin{aligned} p_j(xy \to yx) - p_j(yx \to xy) = &- \sum_{1\leqslant k\leqslant j} \sum_{1\leqslant z\leqslant N} \left(\omega_k(x \to z) - \frac{\det M^{(z)}}{\det M^{(x)}} \omega_k(z \to x) \right) \\ &+ \sum_{1\leqslant k\leqslant j} \sum_{1\leqslant z\leqslant N} \left(\omega_k(y \to z) - \frac{\det M^{(z)}}{\det M^{(y)}} \omega_k(z \to y) \right), \end{aligned} \tag{30}$$

where single-site weights are disappeared. If and only if (30) is satisfied, the model has the stationary product measure with homogeneous single-site weights.

4.2. Case of homogeneous D and p

In this subsection, keeping the single-site weights homogeneous, we also set the rate characterizing particle exchange to be homogeneous: $p_j(x) = p(x)$. As $u_j(xy)$ is independent of j for each x and y, we find from (15) that

$$v_j(1) = v_j(2) = \cdots = v_j(N) \tag{31}$$

and

$$\sum_{1\leqslant z\leqslant N} D(x)\omega_j(x \to z) - \sum_{1\leqslant z\leqslant N} D(z)\omega_j(z \to x) = K D(x) \tag{32}$$

for $2 \leqslant j \leqslant L-1$, where K is the value of (31). Summing over $1 \leqslant x \leqslant N$, we find $K = 0$ and thus

$$\sum_{1\leqslant z\leqslant N} D(x)\omega_j(x \to z) = \sum_{1\leqslant z\leqslant N} D(z)\omega_j(z \to x), \tag{33}$$

for $2 \leqslant j \leqslant L-1$. This means that the total flux of x which is induced by attachment and detachment is zero at every sites except for both the ends. In (25), the terms of $2 \leqslant k \leqslant L-1$ cancel out:

$$D(x)\sum_{k=1,L}\sum_{1\leqslant z\leqslant N} \omega_k(x \to z) - \sum_{k=1,L}\sum_{1\leqslant z\leqslant N} D(z)\omega_k(z \to x) = 0. \tag{34}$$

and the matrix M and single-site weights only depend on the parameters of both the ends. Then the right-hand side of (30) contains only the parameters of both the ends:

$$\begin{aligned} p(xy \to yx) - p(yx \to xy) = &- \sum_{1\leqslant z\leqslant N}\left(\omega_1(x \to z) - \frac{\det M^{(z)}}{\det M^{(x)}}\omega_1(z \to x)\right) \\ &+ \sum_{1\leqslant z\leqslant N}\left(\omega_1(y \to z) - \frac{\det M^{(z)}}{\det M^{(y)}}\omega_1(z \to y)\right). \end{aligned} \tag{35}$$

In the simplest case where $N = 2$, the necessary and sufficient condition (35) is simplified as

$$p(21 \to 12) - q(12 \to 21) = \frac{(\alpha+\beta+\gamma+\delta)(\alpha\beta-\gamma\delta)}{(\alpha+\delta)(\beta+\gamma)}, \tag{36}$$

and (34) as

$$\omega_j(1 \to 2)D(1) = \omega_j(2 \to 1)D(2) \quad (\text{for } 2 \leqslant j \leqslant j-1), \tag{37}$$

where

$$\alpha = \omega_1(1 \to 2), \qquad \beta = \omega_L(2 \to 1), \qquad \gamma = \omega_1(2 \to 1), \qquad \delta = \omega_L(1 \to 2), \tag{38}$$

$$D(1) = (\beta+\gamma)c, \qquad D(2) = (\alpha+\delta)c \tag{39}$$

with a constant c. The constraint (36) is just the same as for the product measure (or the matrix product state with one-dimensional matrices) in the one-species ASEP with injection and extraction of particles at both the ends [4]. The single-site weights (39) are the same as in that case. The constraint (37) is just the detailed balance condition. In [18, 19], exact stationary-state solution to two-species ASEPs with non-conserving dynamics on a ring is studied. In these papers, the stationary-state solutions are obtained as matrix product forms and the non-conserving dynamics satisfy the detailed balance condition, which is similar to (37).

4.3. *Two-segment case*

Let us return to inhomogeneous p_js and consider in the case where the single-site weights are given as

$$D_j(x) = \begin{cases} d_1(x) & 1 \leqslant j \leqslant L' \\ d_2(x) & L'+1 \leqslant j \leqslant L. \end{cases} \tag{40}$$

If $d_1 \neq d_2$, the condition (19) must be satisfied for $j = L'$. Thus, we find the detailed balance condition at the junction of the two segments:

$$p_{L'}(xy \to yx)d_1(x)d_2(y) = p_{L'}(yx \to xy)d_1(y)d_2(x). \tag{41}$$

In a similar way to obtain (33), we also find

$$\sum_{1\leqslant k\leqslant L'} v_k(x)=0,\qquad \sum_{L'+1\leqslant k\leqslant L} v_k(x)=0. \tag{42}$$

We can treat each segment as one chain with homogeneous single-site weights as in subsection 4.1. The same calculation as in subsection 4.1 yields

$$d_1(x)=\det M_1^{(x)},\qquad d_2(x)=\det M_2^{(x)} \tag{43}$$

with $N\times N$ matrices M_1 and M_2 whose elements are

$$(M_1)_{aa}=\sum_{1\leqslant z\leqslant N}\sum_{1\leqslant k\leqslant L'}\omega_k(a\to z),\qquad (M_2)_{aa}=\sum_{1\leqslant z\leqslant N}\sum_{L'+1\leqslant k\leqslant L}\omega_k(a\to z), \tag{44}$$

$$(M_1)_{ab}=-\sum_{1\leqslant k\leqslant L'}\omega_k(b\to a),\qquad (M_2)_{ab}=-\sum_{L'+1\leqslant k\leqslant L}\omega_k(b\to a)\quad (a\neq b), \tag{45}$$

and

$$p_j(xy\to yx)-p_j(yx\to xy)=-\sum_{f\leqslant k\leqslant j}\sum_{1\leqslant z\leqslant N}\left(\omega_k(x\to z)-\frac{\det M_g^{(z)}}{\det M_g^{(x)}}\omega_k(z\to x)\right)$$
$$+\sum_{f\leqslant k\leqslant j}\sum_{1\leqslant z\leqslant N}\left(\omega_k(y\to z)-\frac{\det M_g^{(z)}}{\det M_g^{(y)}}\omega_k(z\to y)\right), \tag{46}$$
$$p_{L'}(xy\to yx)\det M_1^{(x)}\det M_2^{(y)}=p_{L'}(yx\to xy)\det M_1^{(y)}\det M_2^{(x)},$$

where $f=1$ and $g=1$ for $1\leqslant j\leqslant L'-1$ and $f=L'$ and $g=2$ for $L'+1\leqslant j\leqslant L$. This is the necessary and sufficient condition for the stationary product measure in the two-segment case.

5. Conclusion and discussion

We have obtained the necessary and sufficient condition for the multi-species ASEP with attachment and detachment of particles to have the stationary product measure. The condition or restriction (17) was derived by calculating two types of elements of $Q[P]$. One of them corresponds to the configuration $(x,\ldots,x)$ and the other one to $(x,\ldots,x,y,x,\ldots,x)$. We showed that the condition (17) is also a sufficient condition. If the single-site weight is homogeneous, it is given by the determinant of the matrix whose elements consist of the parameters of the attachment and the detachment.

It is natural for us to be tempted to apply this idea to the model in other geometries, such as a ring and higher-dimensional lattices. There exists, however, a necessary condition obtained by calculating these two types of elements, which is not a sufficient one.

Let us consider, for example, the model on an $L_1\times L_2$ two-dimensional lattice with the following rules (see figure 2). The particles exchange their positions at each bond between the site (i,j) and the site $(i,j+1)$ as

$$xy\quad\Longrightarrow\quad yx\qquad\text{with a rate}\quad p_{ij}(xy\to yx). \tag{47}$$

Similarly they do at each bond between the site (i,j) and the site $(i+1,j)$ as

$$\begin{matrix}x\\y\end{matrix}\quad\Longrightarrow\quad\begin{matrix}y\\x\end{matrix}\qquad\text{with a rate}\quad q_{ij}\begin{pmatrix}x & & y\\ & \to & \\ y & & x\end{pmatrix}. \tag{48}$$

Detachment of x and attachment of y take place simultaneously at each site (i,j) as

$$x\quad\Longrightarrow\quad y\qquad\text{with a rate}\quad \omega_{ij}(x\to y) \tag{49}$$

for $1\leqslant x,y\leqslant N, x\neq y$.

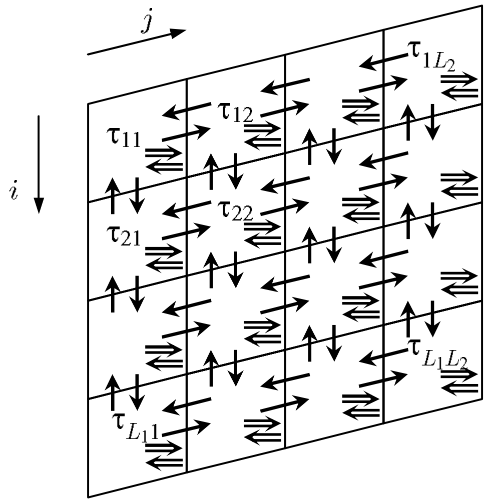

Figure 2. Multi-species ASEP with attachment and detachment in two dimension.

If the model has the stationary product measure

$$P\begin{pmatrix}\tau_{11} & \cdots & \tau_{1L_2}\\ \vdots & \ddots & \vdots \\ \tau_{L_11} & \cdots & \tau_{L_1L_2}\end{pmatrix} = \frac{1}{Z}\prod_{\substack{1\leqslant i\leqslant L_1\\ 1\leqslant j\leqslant L_2}} D_{ij}(\tau_{ij}), \tag{50}$$

the parameters should satisfy

$$\begin{aligned}&\sum_{\substack{1\leqslant i\leqslant L_1\\ 1\leqslant j\leqslant L_2}} v_{ij}(x) = 0,\\ &u_{ij-1}(xy) + u_{ij}(yx) + u'_{i-1j}\begin{pmatrix}x\\ y\end{pmatrix} + u'_{ij}\begin{pmatrix}y\\ x\end{pmatrix} = v_{ij}(x) - v_{ij}(y),\end{aligned} \tag{51}$$

for $1 \leqslant x, y \leqslant N, x \neq y$, where

$$u_{ij}(xy) = p_{ij}(xy \to yx) - p_{ij}(yx \to xy)\frac{D_{ij}(y)D_{ij+1}(x)}{D_{ij}(x)D_{ij+1}(y)}, \tag{52}$$

$$u'_{ij}\begin{pmatrix}x\\ y\end{pmatrix} = q_{ij}\begin{pmatrix}x & & y\\ & \to & \\ y & & x\end{pmatrix} - q_{ij}\begin{pmatrix}y & & x\\ & \to & \\ x & & y\end{pmatrix}\frac{D_{ij}(y)D_{i+1j}(x)}{D_{ij}(x)D_{i+1j}(y)}, \tag{53}$$

$$v_{ij}(x) = \sum_{1\leqslant z\leqslant N}\omega_{ij}(x\to z) - \sum_{1\leqslant z\leqslant N}\omega_{ij}(z\to x)\frac{D_{ij}(z)}{D_{ij}(x)}. \tag{54}$$

Counterexamples, however, can be shown. Namely we can find some sets of rate p_{ij}, q_{ij} and ω_{ij} which satisfy the condition (51) and do not let the form (50) be the stationary-state solution.

In one dimension, what we should do next is to apply the idea to more general reaction–diffusion models where the following interactions of particles are allowed:

$$xy \quad \Longrightarrow \quad zw \tag{55}$$

for $1 \leqslant x, y, z, w \leqslant N$. To determine whether the necessary condition obtained by using the idea is also a sufficient condition or not will be a future work.

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Appendix

Let us prove here formula (22). Assuming that (22) is true for i, we can show that it is also true for $i+1$:

$$Q_{i+1}[\hat{P}](\tau_1,\ldots,\tau_L) \tag{A.1}$$

$$\begin{aligned}
&= Q_i[\hat{P}](\tau_1,\ldots,\tau_L)\\
&\quad + p_{i+1}(\tau_{i+1}\tau_{i+2}\to\tau_{i+2}\tau_{i+1})\hat{P}(\tau_1,\ldots,\tau_L)\\
&\quad - p_{i+1}(\tau_{i+2}\tau_{i+1}\to\tau_{i+1}\tau_{i+2})\hat{P}(\tau_1,\ldots,\tau_i,\tau_{i+2},\tau_{i+1},\tau_{i+3},\ldots,\tau_L)\\
&\quad + \sum_{1\leqslant z\leqslant N}\omega_{i+1}(\tau_{i+1}\to z)\hat{P}(\tau_1,\ldots,\tau_L)\\
&\quad - \sum_{1\leqslant z\leqslant N}\omega_{i+1}(z\to\tau_{i+1})\hat{P}(\tau_1,\ldots,\tau_i,z,\tau_{i+2},\ldots,\tau_L)
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&= \sum_{1\leqslant\ell\leqslant i} v_\ell(\tau_{i+1})D_1(\tau_1)\cdots D_L(\tau_L)\\
&\quad + u_{i+1}(\tau_{i+1}\tau_{i+2})D_1(\tau_1)\cdots D_L(\tau_L) + v_{i+1}(\tau_{i+1})D_1(\tau_1)\cdots D_L(\tau_L)
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
&= D_1(\tau_1)\cdots D_L(\tau_L)\sum_{1\leqslant\ell\leqslant i} v_\ell(\tau_{i+1}) + D_1(\tau_1)\cdots D_L(\tau_L)\\
&\quad \times \sum_{1\leqslant\ell\leqslant i+1}(v_\ell(\tau_{i+2}) - v_\ell(\tau_{i+1})) + v_{i+1}(\tau_{i+1})D_1(\tau_1)\cdots D_L(\tau_L)
\end{aligned} \tag{A.4}$$

$$= D_1(\tau_1)\cdots D_L(\tau_L)\sum_{1\leqslant\ell\leqslant i+1} v_\ell(\tau_{i+2}). \tag{A.5}$$

In going from (A.3) to (A.4), we used the constraint (17).

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